LAGRANGIAN POINT-BASED METHODS

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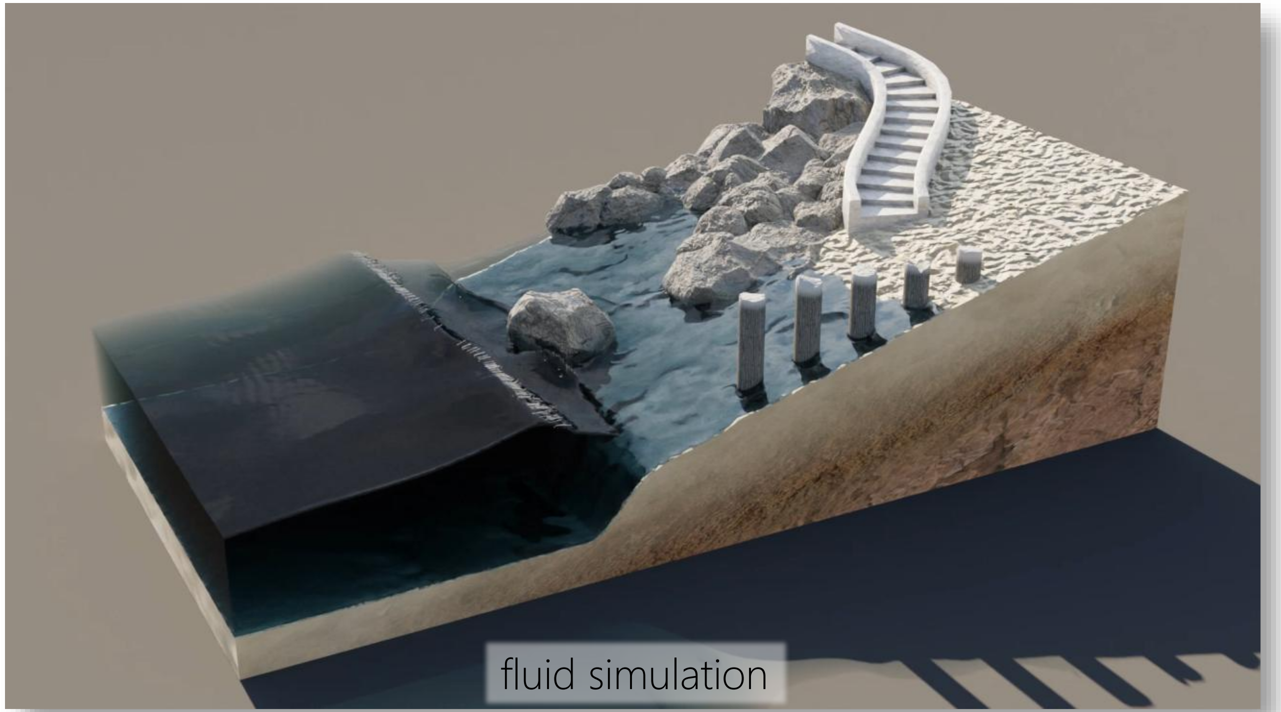
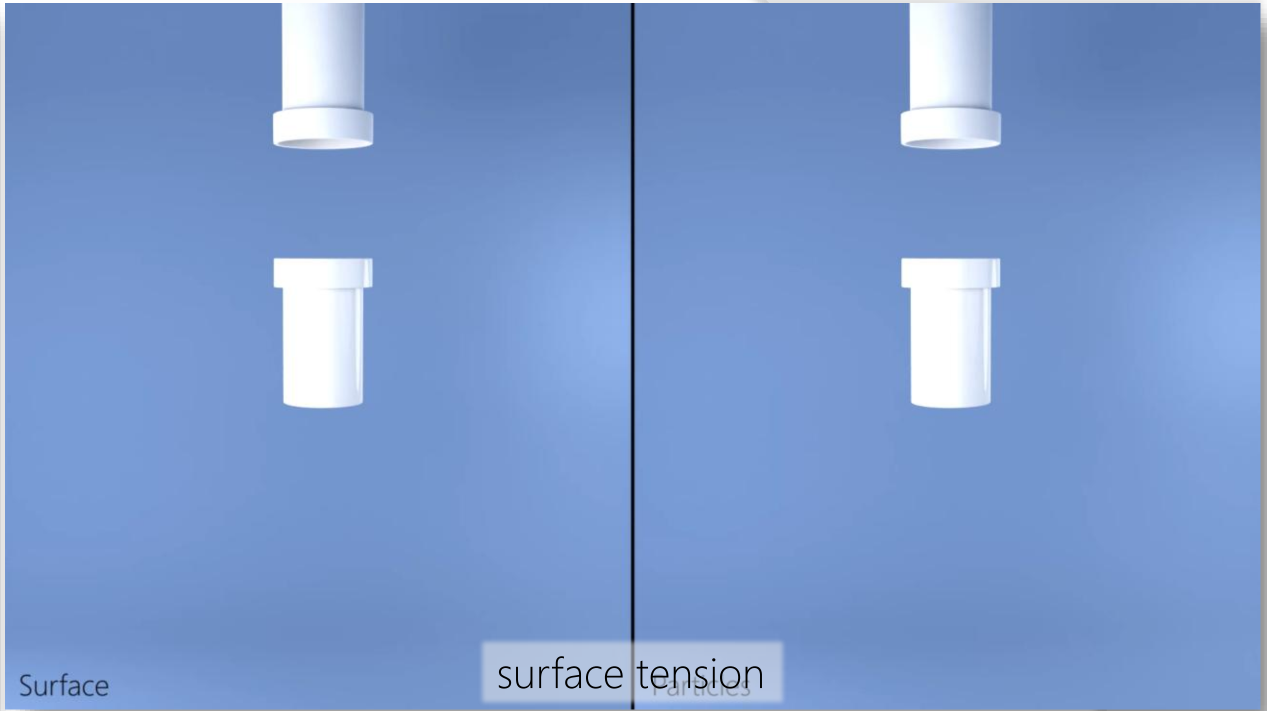
Content

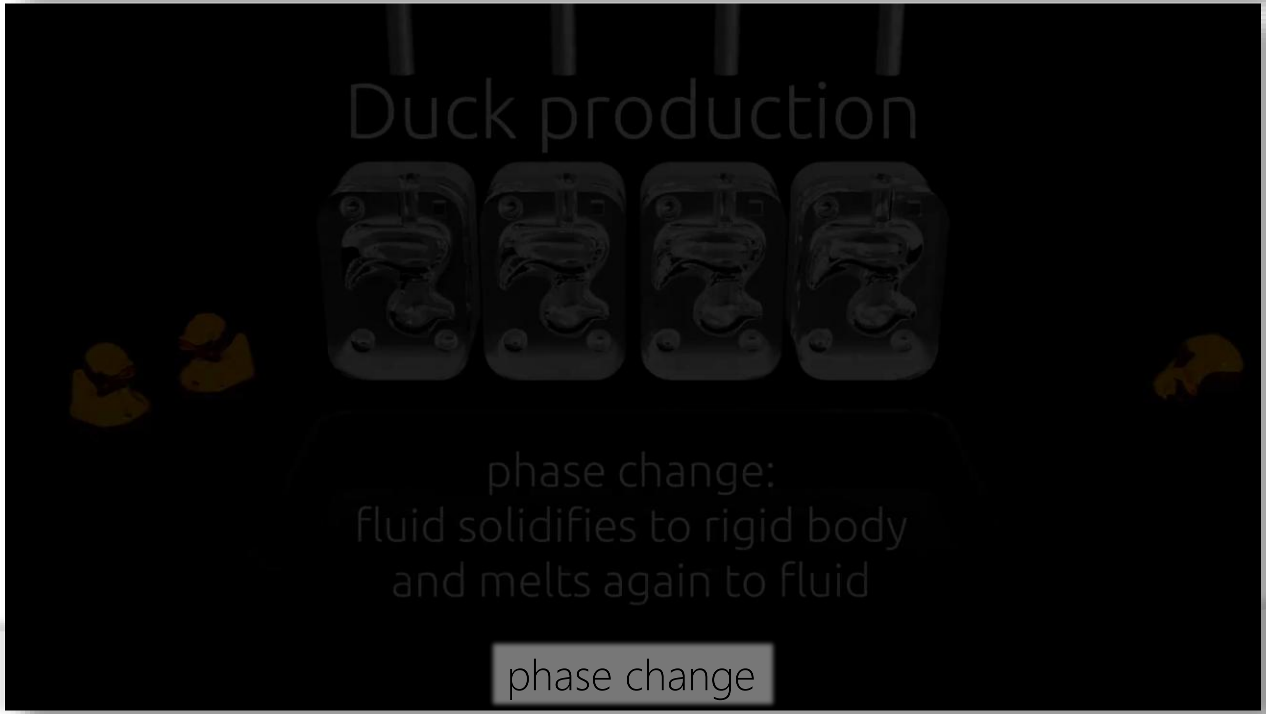
- Motivation
- Smoothed Particle Hydrodynamics
- Materials and Phenomena
- Conclusion



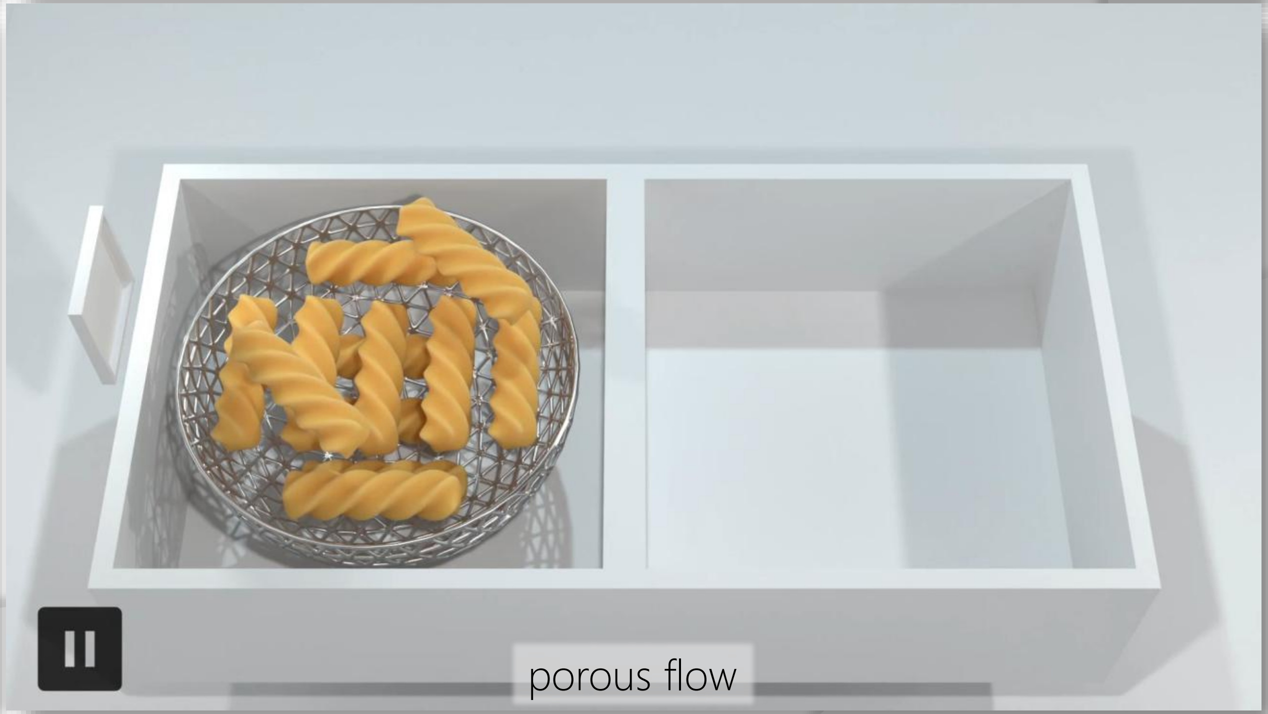
fluid simulation



surface tension



phase change



porous flow

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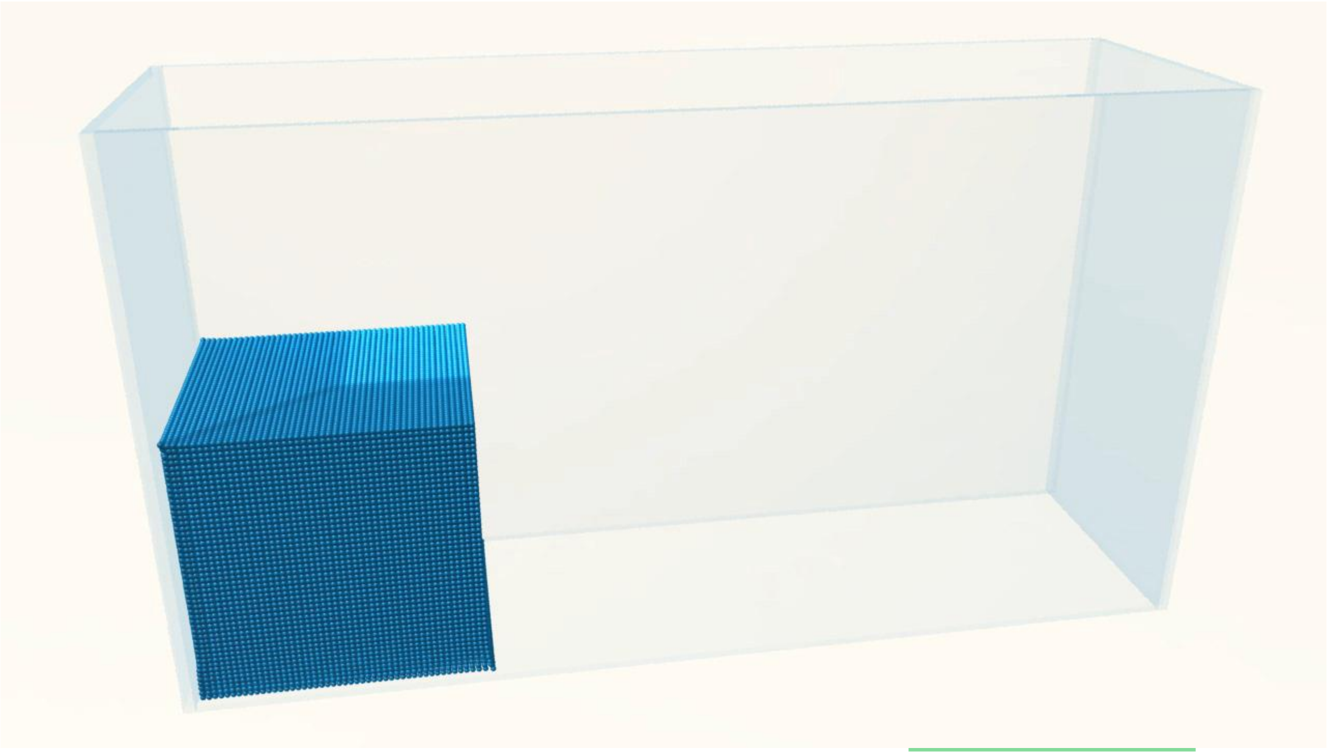
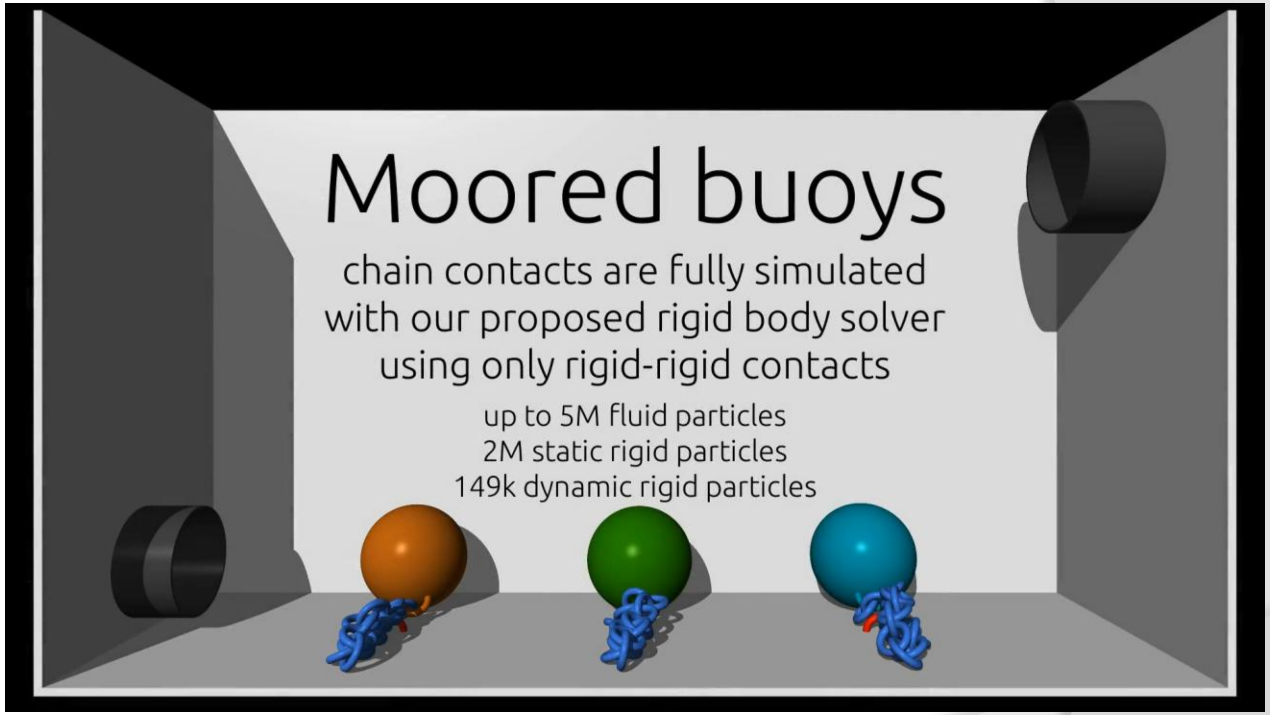
Smoothed Particle Hydrodynamics (SPH)

- SPH is a mesh-free method for the discretization of functions and partial differential operators.
  - Functions are discretized into samples equipped with kernel function W
  - Approximates/discretizes differential operators



Smoothed Particle Hydrodynamics (SPH)

- Useful to simulate continuum media
  - conservational properties
  - greatly handles topological changes
  - algorithms parallelize well
  - good for advection-type/transport problems
  - supports fluids, deformable solids, rigid bodies, highly viscous materials, granular materials, and snow



Dirac-Delta Distribution

- The Dirac-Delta distribution is a function defined as

$$\delta(x) = \begin{cases} \infty & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{+\infty} \delta(x) dx = 1$$

Dirac-Delta Identity

- Given a function $f : \Omega \rightarrow \mathbb{R}^n$ that maps a position vector $\mathbf{x}$ in the domain $\Omega \subset \mathbb{R}^3$ to a scalar or vector value, it can be rewritten using the Dirac-delta identity:

$$f(\mathbf{x}) = \int_\Omega f(\mathbf{x}^*)\ \delta(\mathbf{x} - \mathbf{x}^*) d\mathbf{x}^*$$

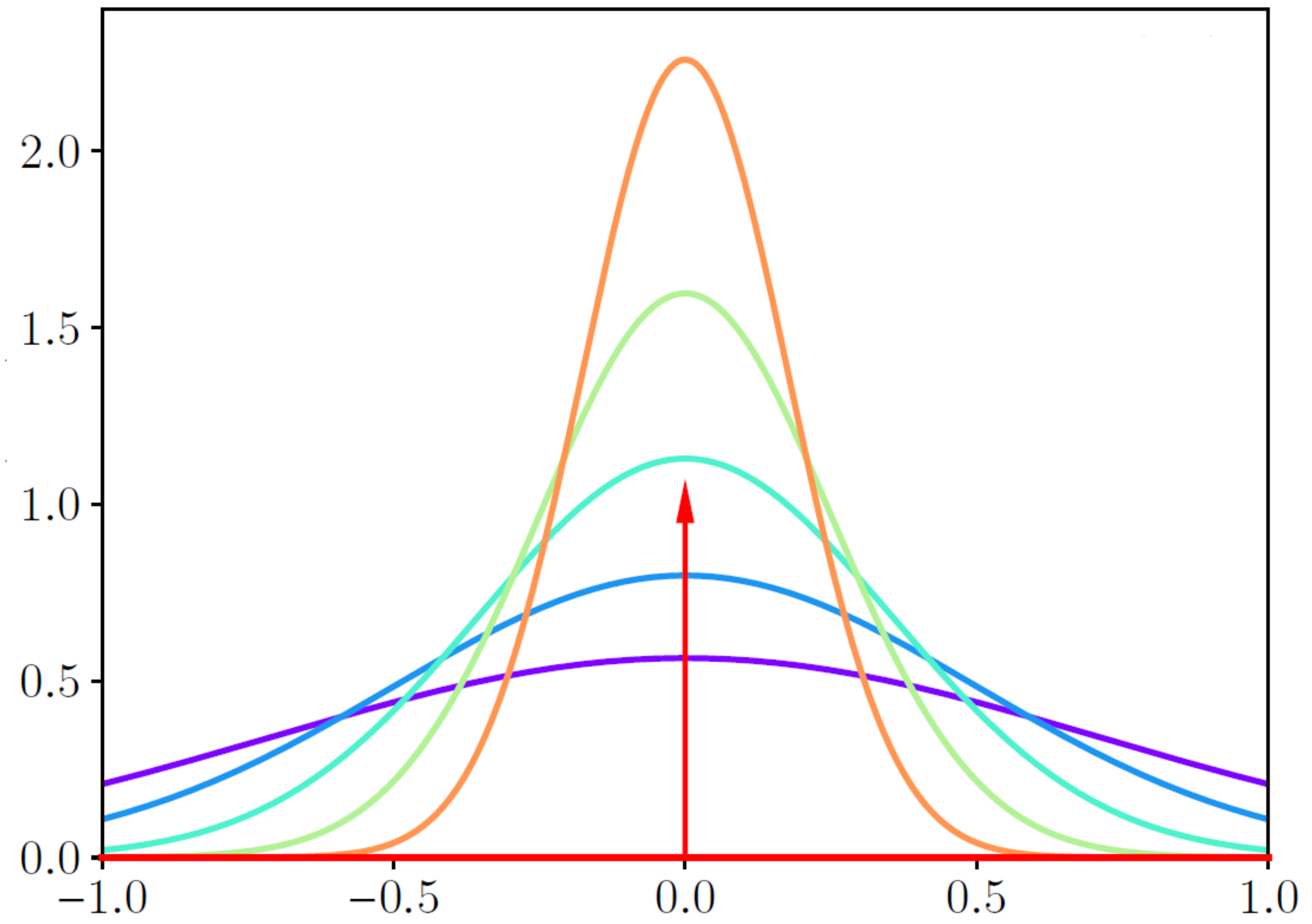
Kernel Function

- Approximation by replacing the delta distribution with a kernel function W with compact support:

$$
\begin{aligned}
f(\mathbf{x}) &= \int_{\Omega} f(\mathbf{x}^*)\ \delta(\mathbf{x} - \mathbf{x}^*) d\mathbf{x}^* \\
&\approx \int_{\mathcal{D}_{\mathbf{x}}} f(\mathbf{x}^*)\ W(\mathbf{x} - \mathbf{x}^*, h) d\mathbf{x}^*
\end{aligned}
$$

- $\mathcal{D}_{\mathbf{x}}$ represents the spherical domain around $\mathbf{x}$ where the kernel is not zero.

Kernel Function



Conditions of a Kernel Function

- Normalization condition: $\int W(\mathbf{x}-\mathbf{x}^*, h)d\mathbf{x}^* = 1$
- Symmetry condition: $W(\mathbf{x}-\mathbf{x}^*, h) = W(\mathbf{x}^*-\mathbf{x}, h)$
- Delta function property: $\lim_{h\to 0} W(\mathbf{x}-\mathbf{x}^*, h) = \delta(\mathbf{x}-\mathbf{x}^*)$
- Non-negative condition: $W(\mathbf{x}-\mathbf{x}^*, h) \geq 0$
- Compact condition with support radius r:

$$W(\mathbf{x}-\mathbf{x}^*, h) = 0 \quad \text{if} \quad \|\mathbf{x}-\mathbf{x}^*\| > r$$

Kernel Function - Example

- Cubic spline kernel:

$$W(q) = \alpha \begin{cases} \frac{2}{3} - q^2 + \frac{1}{2}q^3 & 0 \leq q < 1 \\ \frac{1}{6}(2-q)^3 & 1 \leq q < 2 \\ 0 & q \geq 2 \end{cases}$$

$$q = \frac{\|\mathbf{x}_i - \mathbf{x}_j\|}{h}$$

- 1D: $\alpha = \frac{1}{h}$, 2D: $\alpha = \frac{15}{7\pi h^2}$, 3D: $\alpha = \frac{3}{2\pi h^3}$

Smoothed Particle Hydrodynamics

- Finally, the integral

$$f(\mathbf{x}) \approx \int_{\mathcal{D}_{\mathbf{x}}} f(\mathbf{x}^*)\, W(\mathbf{x} - \mathbf{x}^*, h) d\mathbf{x}^*$$

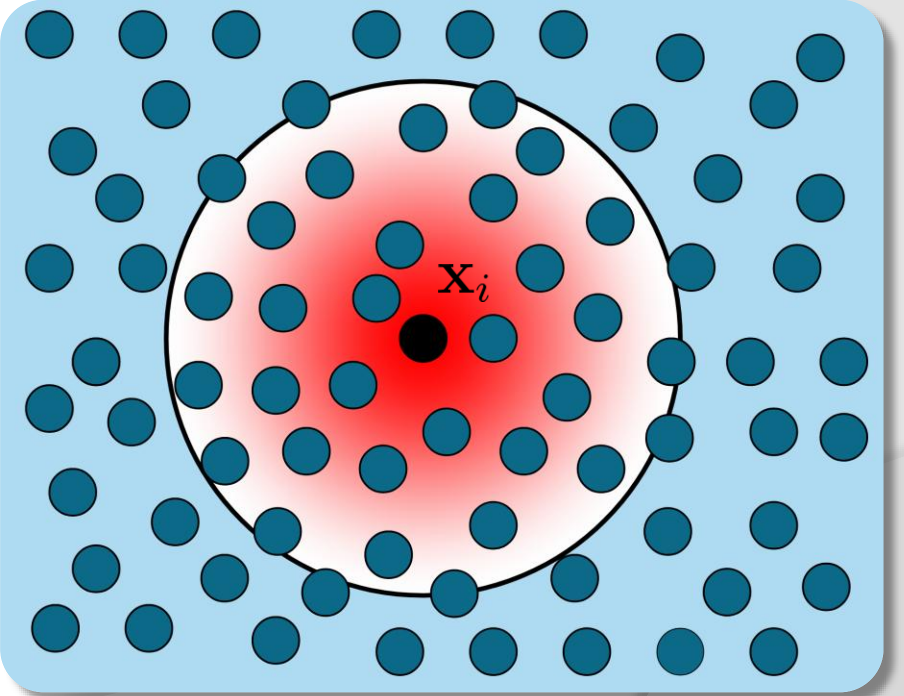
can be discretized into a sum over particles with positions $\mathbf{x}_j$ and volumes V_j :

$$f(\mathbf{x}) \approx \sum_j V_j f(\mathbf{x}_j) W(\mathbf{x} - \mathbf{x}_j, h).$$

Smoothed Particle Hydrodynamics

- In the SPH formulation a quantity A_i is approximated by the quantities A_j at the neighboring locations $\mathbf{x}_j$:

$$A_i \approx \sum_j \frac{m_j}{\rho_j} A_j W_{ij}$$



$$W_{ij} = W(\mathbf{x}_i - \mathbf{x}_j, h)$$

Smoothed Particle Hydrodynamics

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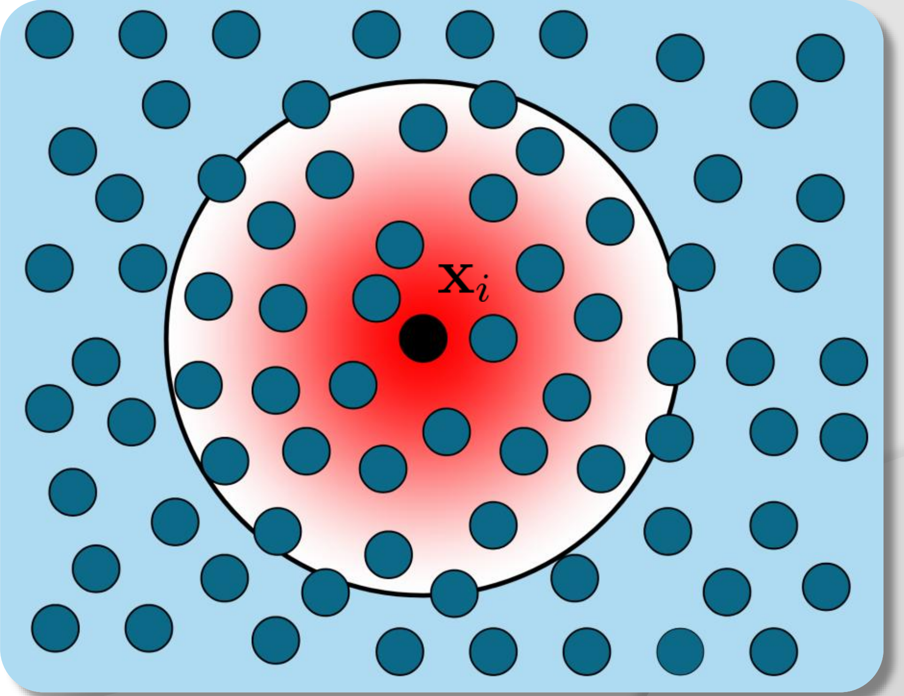


- Density:

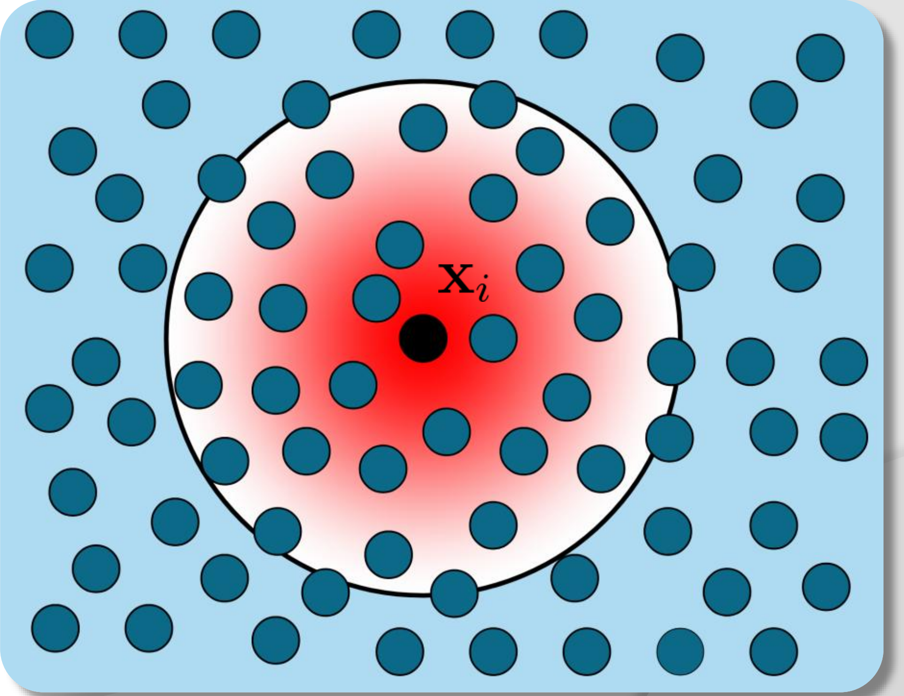
$$\rho_i \approx \sum_j \frac{m_j}{\rho_j} \rho_j W_{ij} = \sum_j m_j W_{ij}$$

$$W_{ij} = W(\mathbf{x}_i - \mathbf{x}_j, h)$$

Smoothed Particle Hydrodynamics

- In the SPH formulation a quantity A_i is approximated by the quantities A_j at the neighboring locations $\mathbf{x}_j$:

$$A_i \approx \sum_j \frac{m_j}{\rho_j} A_j W_{ij}$$



- Gradient:

$$\nabla A_i = \frac{\partial A_i}{\partial \mathbf{x}_i} \approx \sum_j \frac{m_j}{\rho_j} A_j \nabla W_{ij}$$

$$W_{ij} = W(\mathbf{x}_i - \mathbf{x}_j, h)$$

Discrete Differential Operators

- Improved variants:

$$\nabla A_i \approx \rho_i \sum_j m_j \left(\frac{A_i}{\rho_i^2} + \frac{A_j}{\rho_j^2} \right) \nabla W_{ij}$$

preserves linear and angular momentum when used for pressure forces

$$\nabla^2 \mathbf{A}_i \approx 2(d+2) \sum_j \frac{m_j}{\rho_j} \frac{(\mathbf{A}_i - \mathbf{A}_j) \cdot \mathbf{x}_{ij}}{\|\mathbf{x}_{ij}\|^2 + 0.01h^2} \nabla W_{ij}$$

more robust since second derivative of kernel is avoided

$$\mathbf{x}_{ij} = \mathbf{x}_i - \mathbf{x}_j$$

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Lets solve the Navier-Stokes equations:

$$\rho\frac{D\mathbf{v}}{Dt} = -\nabla \boxed{p} + \mu\nabla^2\mathbf{v} + \mathbf{f}^{\text{ext}}$$

- Pressure can be determined by Equation of State (EOS)

$$p_i = \kappa\left(\rho_i - \rho_0\right)$$

κ is a stiffness coefficient.

Pressure Force

$$\rho \frac{D\mathbf{v}}{Dt} = \boxed{-\nabla p} + \mu \nabla^2 \mathbf{v} + \mathbf{f}^{\text{ext}}$$

- Pressure force $\mathbf{f}^{\mathrm{p}}$:

$$\nabla A_i \approx \rho_i \sum_j m_j \left(\frac{A_i}{\rho_i^2} + \frac{A_j}{\rho_j^2} \right) \nabla W_{ij}$$

$$\nabla p_{\boldsymbol{i}} \approx \boldsymbol{\rho_i} \sum_{\boldsymbol{j}} \boldsymbol{m_j} \left(\frac{p_i}{\boldsymbol{\rho_i^2}} + \frac{p_j}{\rho_j^2} \right) \nabla W_{\boldsymbol{ij}}$$

Viscosity

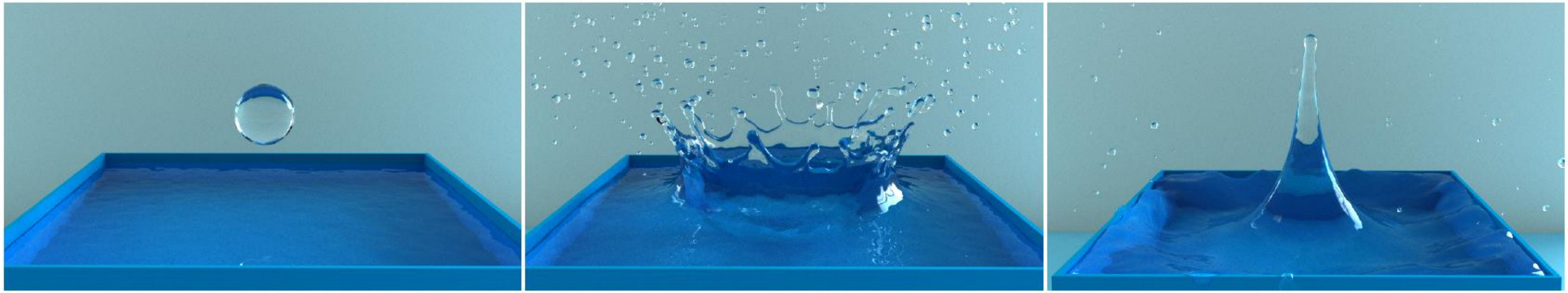
$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla p + \boxed{\mu \nabla^2 \mathbf{v}} + \mathbf{f}^{\text{ext}}$$

- Viscous force:

$$\nabla^2 \mathbf{v}_i \approx 2(d+2) \sum_j \frac{m_j}{\rho_j} \frac{(\mathbf{v}_i - \mathbf{v}_j) \cdot \mathbf{x}_{ij}}{\|\mathbf{x}_{ij}\|^2 + 0.01h^2} \nabla W_{ij}$$

Surface Tension

- The surface tension of a fluid has to be considered for realistic results.
- This is especially important for small-scale effects, e.g. when simulating droplets.



N. Akinci, G. Akinci, M. Teschner, "Versatile Surface Tension and Adhesion for SPH Fluids“, SIGGRAPH Asia 2013

Surface Tension

- Surface tension arises due to attractive forces between molecules.
- This can be modeled as a cohesion force:

$$\mathbf{a}_i^{\text{cohesion}} = -\frac{\alpha}{m_i}\sum_j m_j(\mathbf{x}_i - \mathbf{x}_j)W_{ij}$$

M. Becker & M. Teschner, "Weakly compressible
SPH for free surface flows", SCA 2007

Simulation Step

```
for all particles $i$ do
    find neighbors $j$
for all particles $i$ do
    $\rho_i = \sum_j m_j W_{ij}$
    compute $p_i$ using EOS
for all particles $i$ do
    $\mathbf{a}_i^{\text{pressure}} = -\frac{1}{\rho_i} \nabla p_i$
    $\mathbf{a}_i^{\text{viscosity}} = \nu \nabla^2 \mathbf{v}_i$
    $\mathbf{a}_i^{\text{cohesion}} = -\frac{\kappa}{m_i} \sum_j m_j (\mathbf{x}_i - \mathbf{x}_j) W_{ij}$
    $\mathbf{a}_i(t) = \mathbf{a}_i^{\text{pressure}} + \mathbf{a}_i^{\text{viscosity}} + \mathbf{a}_i^{\text{cohesion}} + \mathbf{a}_i^{\text{ext}}$
for all particles $i$ do
    $\mathbf{v}_i(t + \Delta t) = \mathbf{v}_i(t) + \Delta t \mathbf{a}_i(t)$
    $\mathbf{x}_i(t + \Delta t) = \mathbf{x}_i(t) + \Delta t \mathbf{v}_i(t + \Delta t)$
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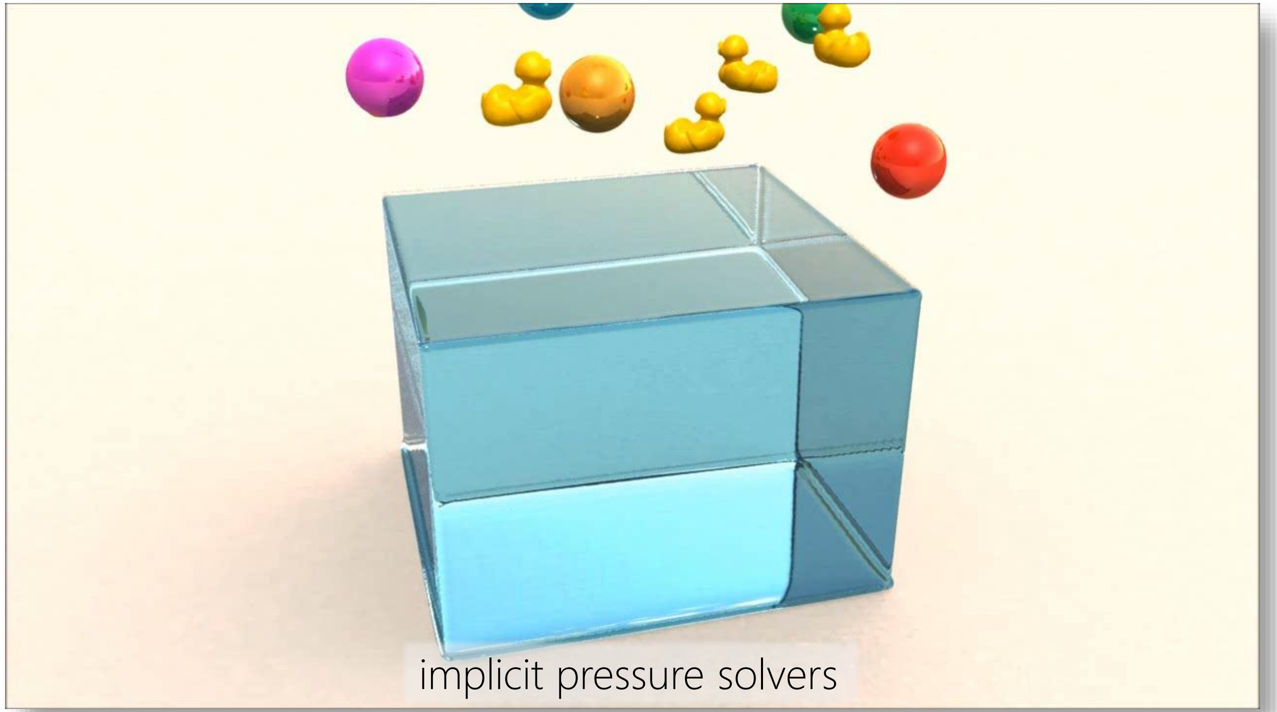
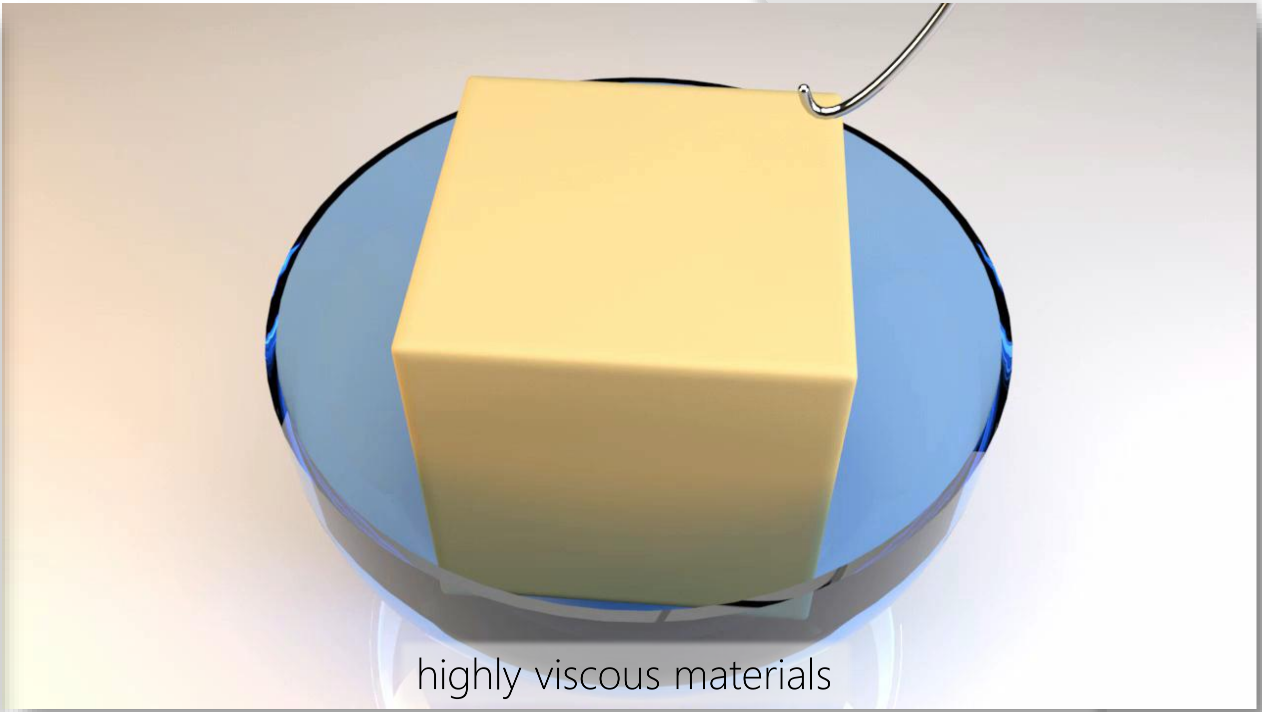
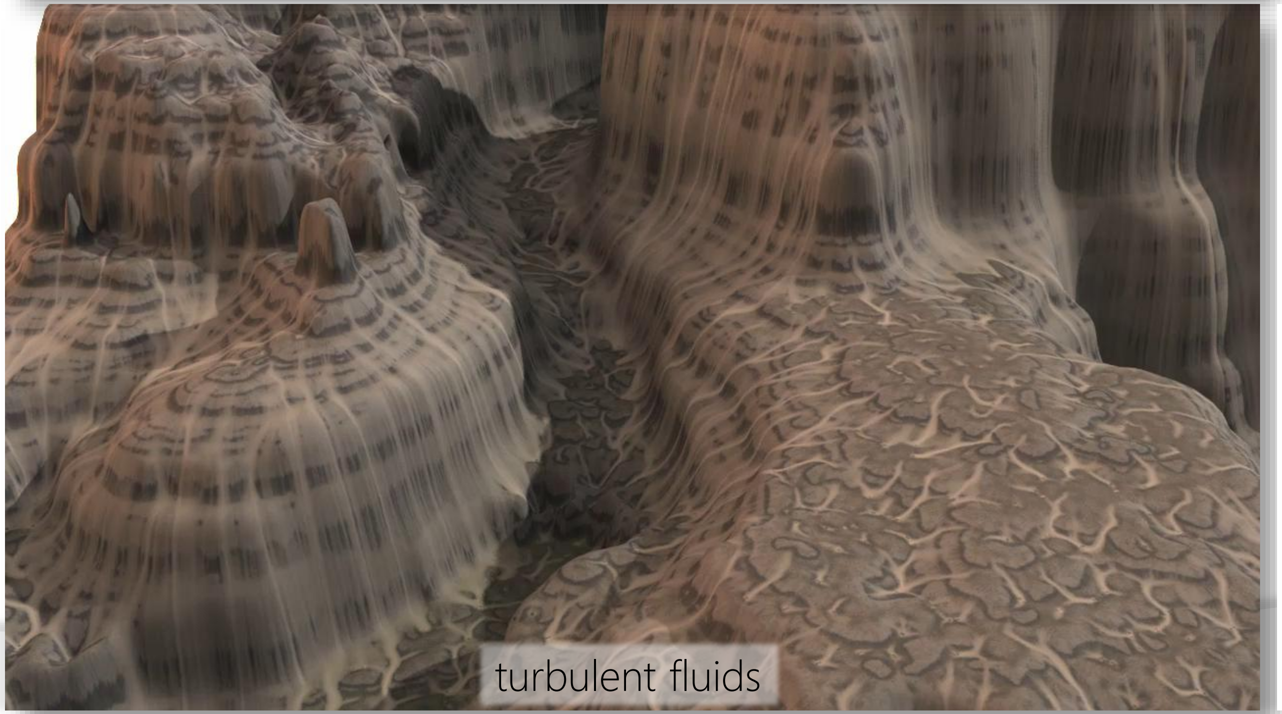


implicit pressure solvers

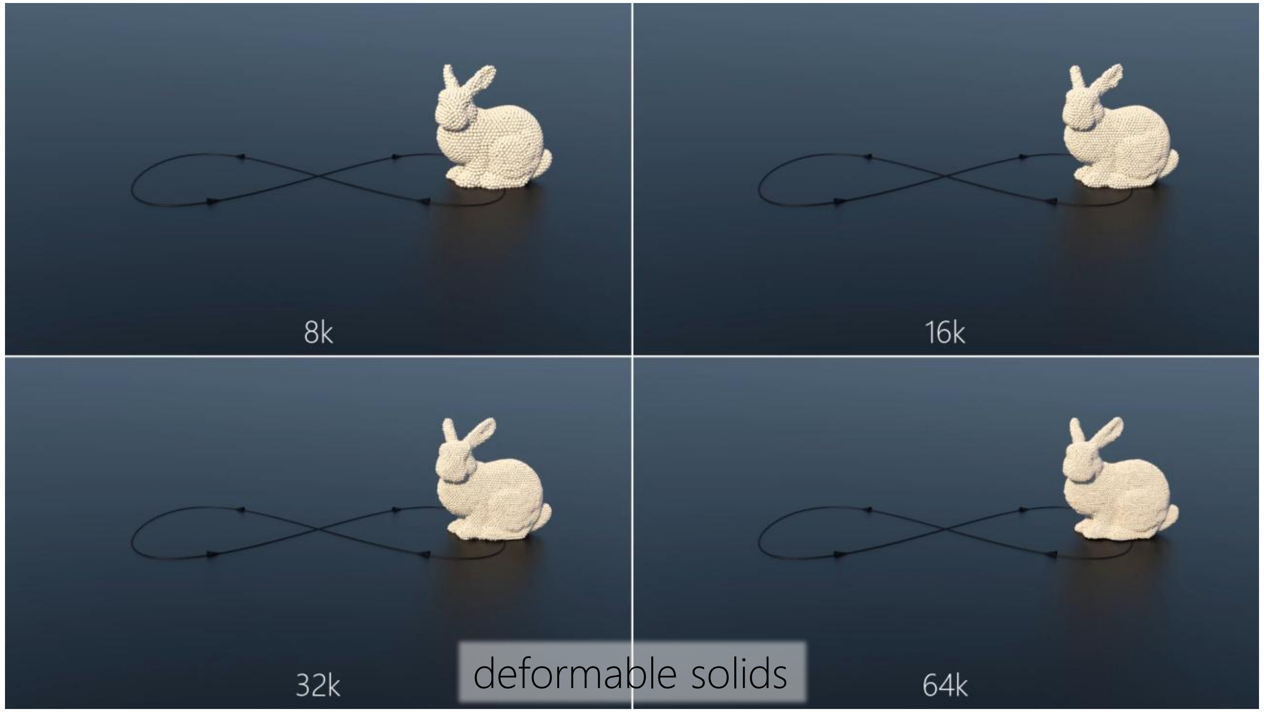


highly viscous materials

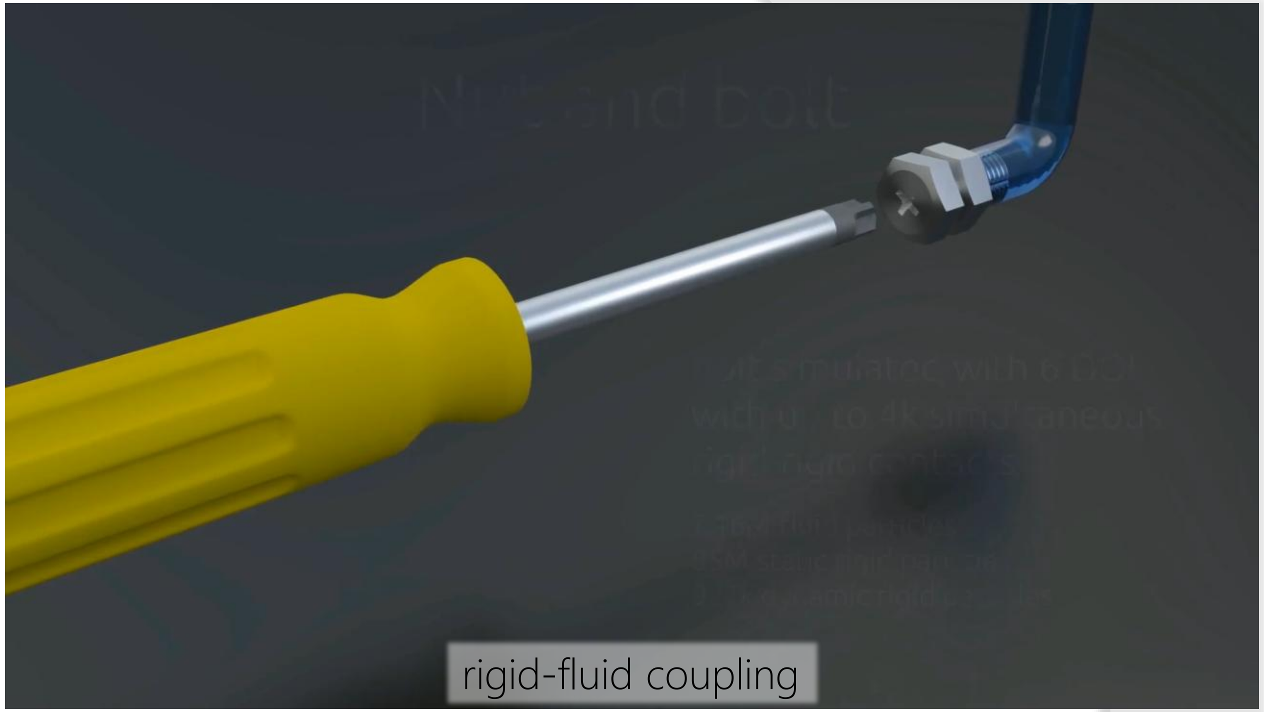


turbulent fluids

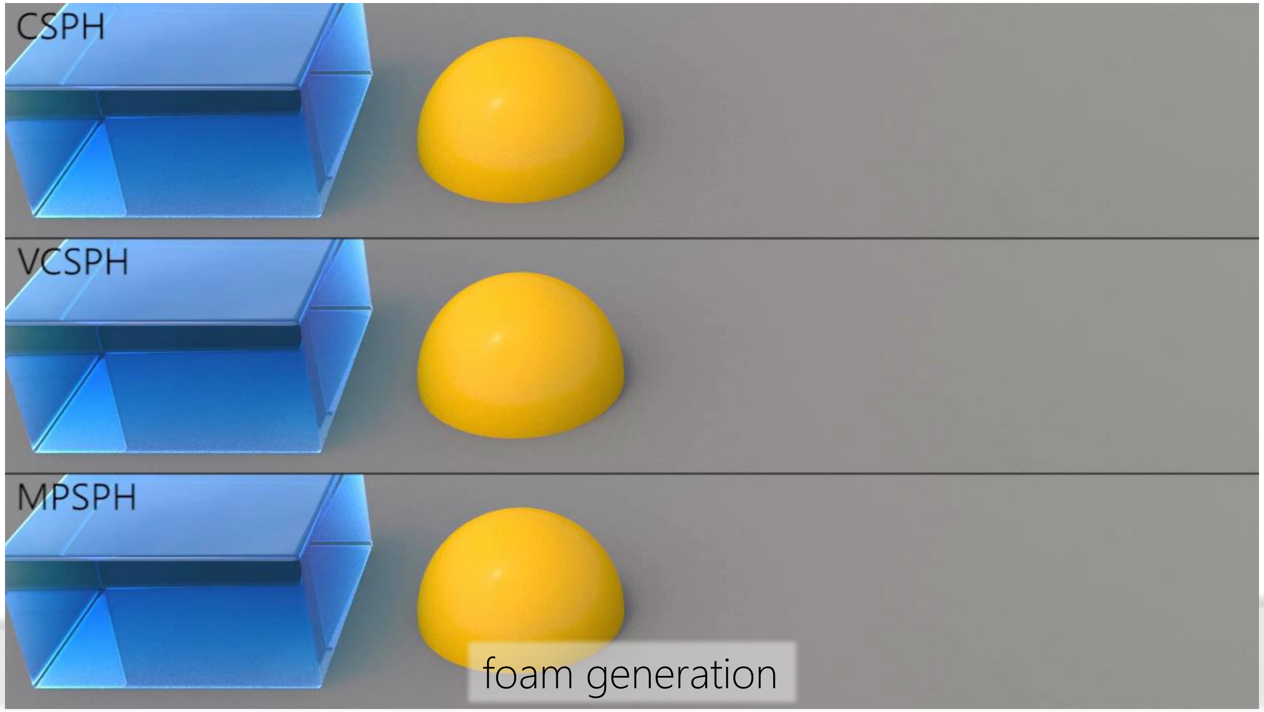
boundary handling



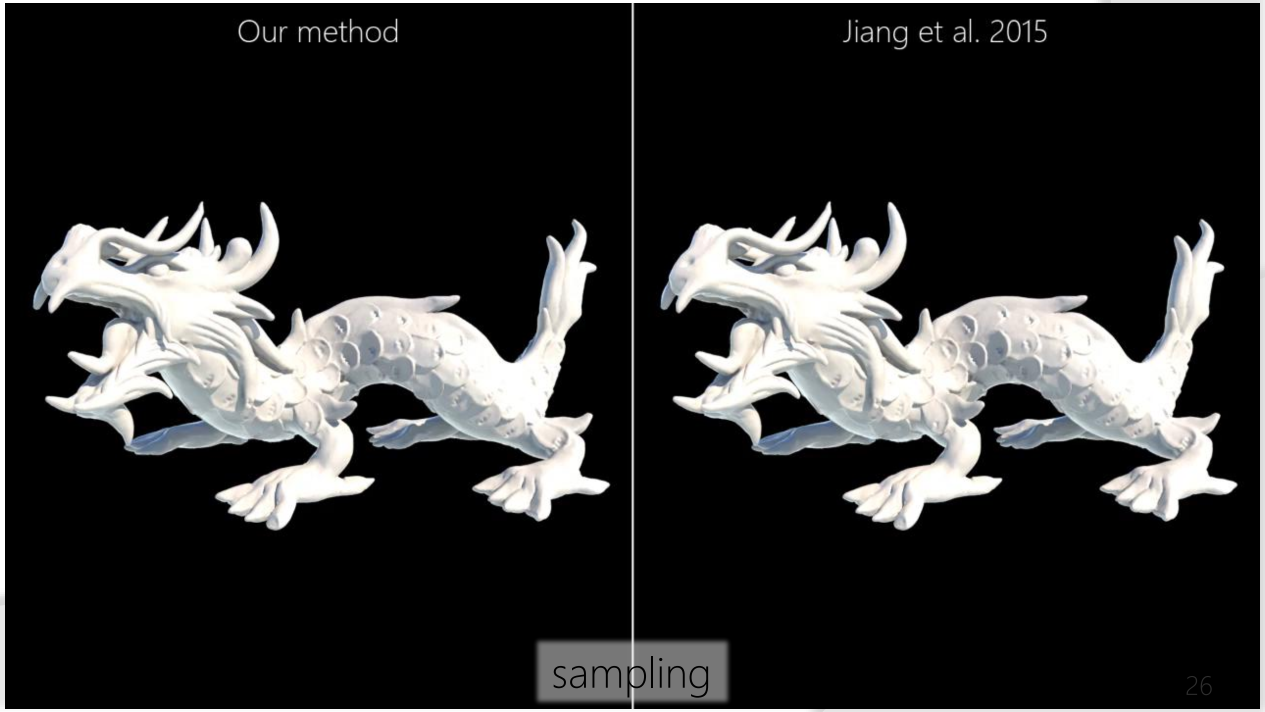
deformable solids



rigid-fluid coupling



foam generation



sampling

Further Research

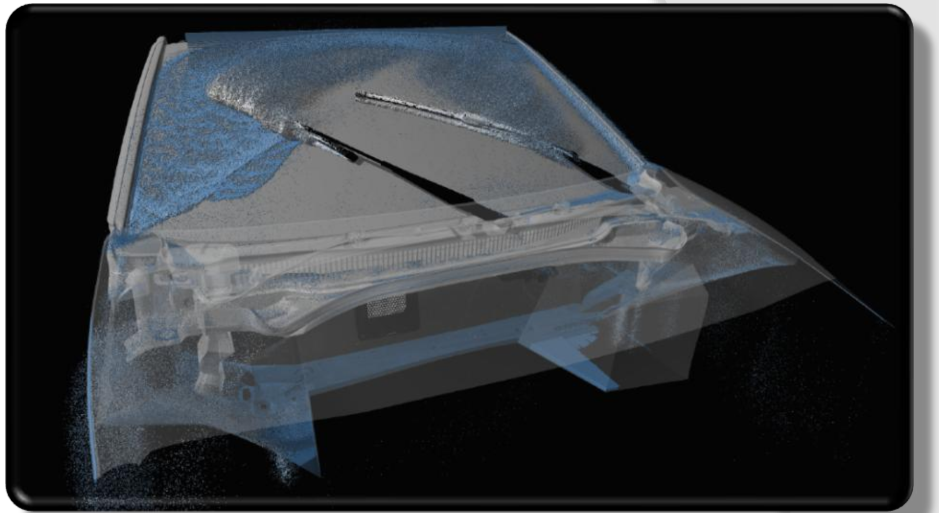
Snow

Gissler et al., SIGGRAPH 2020

Granular materials

Ihmsen et al., C&G 2013

Air drag

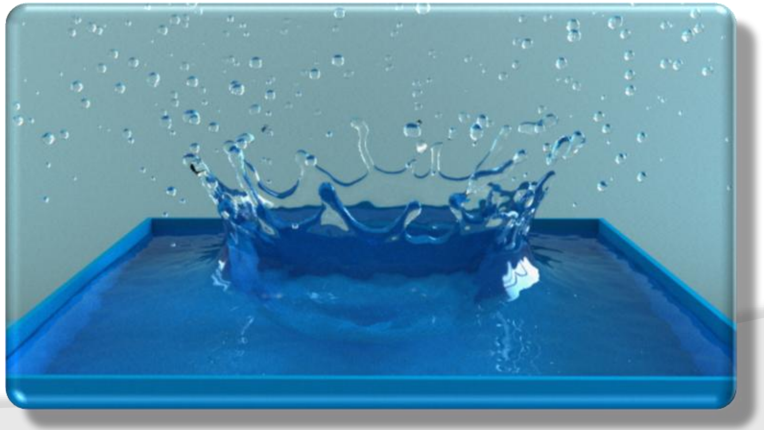


Gissler et al., C&G 2017

Deformable solids

Peer et al., CGF 2017

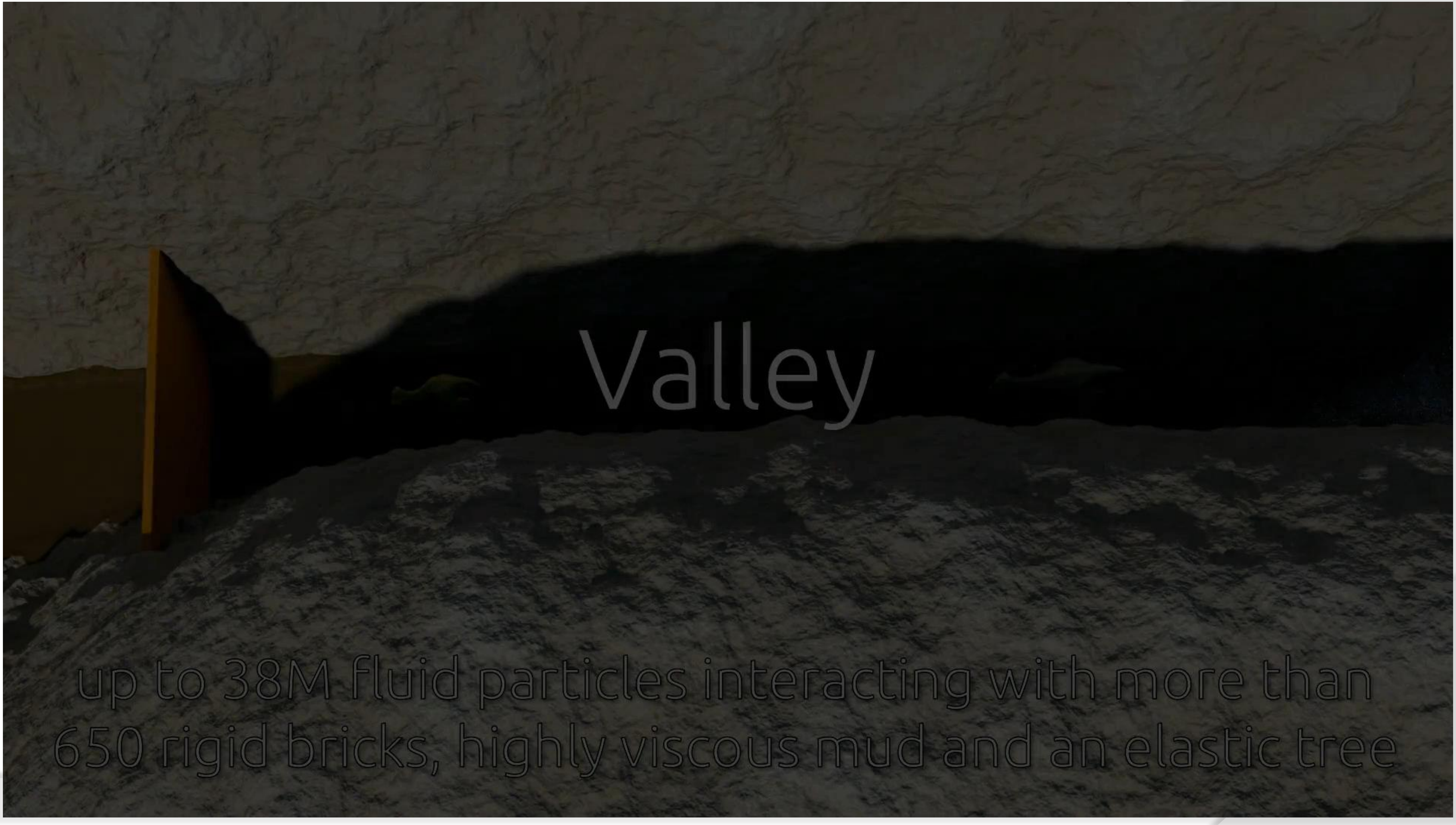
Surface tension



Akinci et al., SIGGRAPH Asia 2013

Data-driven simulation

Ladický et al., ACM TOG 2015



Valley

up to 38M fluid particles interacting with more than 650 rigid bricks, highly viscous mud and an elastic tree

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Conclusion

- SPH has several advantages:
  - supports different material models (e.g., rigids, deformables, fluids)
  - coupling different models is simple
  - supports topology changes
  - enables complex effects like melting, solidification

Thank you for your attention!

https://splishsplash.physics-simulation.org